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2005 J. Phys. A: Math. Gen. 38 2803

(http://iopscience.iop.org/0305-4470/38/12/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2803–2817 doi:10.1088/0305-4470/38/12/020

Nonlinear Kelvin–Helmholtz instability of fluid layers with mass and heat transfer

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Received 12 July 2004, in final form 7 February 2005
Published 9 March 2005
Online at stacks.iop.org/JPhysA/38/2803

Abstract
The nonlinear Kelvin–Helmholtz stability of the interface between the vapour and liquid phases of a fluid is studied when the phases are confined between two parallel lines, and when there is mass and heat transfer across the interface. The method of multiple time expansion is used for the investigation. The evolution of amplitude is shown to be governed by a nonlinear first-order differential equation. The stability criterion is discussed, and the region of stability is displayed graphically.

PACS number: 47.20.Ma

Nomenclature

$F(x, y, t) = 0$	function for the interface
$\mathbf{n}$	outward vector normal to the interface
η	perturbation from the equilibrium interface
$\rho^{(1)}$	density of fluid 1
$\rho^{(2)}$	density of fluid 2
U_1	streaming velocity of fluid 1
U_2	streaming velocity of fluid 2
$\phi^{(1)}$	velocity potential for fluid 1
$\phi^{(2)}$	velocity potential for fluid 2
L	the latent heat
$S(\eta)$	net heat flux from the interface
K_1	heat conductivity of fluid 1
K_2	heat conductivity of fluid 2
h_1	depth of fluid 1
h_2	height of fluid 2
T_0	temperature at $y = 0$
T_1	temperature at $y = -h_1$
T_2	temperature at $y = h_2$
p_1	pressure in fluid 1

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p_2	pressure in fluid 2
σ	surface tension
$A(t_1, t_2)$	amplitude of the perturbation η
ω	frequency of the wave
k	wave number
g	gravity

1. Introduction

In dealing with the flow of two fluids divided by an interface, the problem of interfacial stability is usually studied without taking into account the heat and mass transfer across the interface. The classical Kelvin–Helmholtz stability [1] as well as other variations of the problem, such as the stability of bubble motion in a liquid, all belong to this category. However, there are situations when the effect of mass and heat transfer across the interface should be taken into account in stability discussions. For instance, the phenomenon of boiling accompanies high heat and mass transfer rates which are significant in determining the flow field and the stability of the system.

Hsieh [2] presented a simplified formulation of interfacial flow problem with mass and heat transfer, and studied the problems of Rayleigh–Taylor and Kelvin–Helmholtz stability in the plane geometry. Knowledge of the mechanism of heat and mass transfer across an interface is important in many areas of engineering. Such a mechanism can be noticeable in various industrial processes such as gas absorption, evaporation equipment, and so on.

Hsieh [3] found that from the linearized analysis, when the vapour region is hotter than the liquid region, as is usually so, the effect of mass and heat transfer tends to inhibit the growth of the instability. Thus for the problem of film boiling, the instability would be reduced yet would persist according to linear analysis.

It is clear that such a uniform model based on the linear theory is inadequate to answer the question of whether and how the effect of heat mass would stabilize the system, since the stability criterion remains the same regardless the amount of heat flux of the system, so the nonlinear analysis is needed to answer the question.

Although the nonlinear problem of Rayleigh–Taylor instability of a system of two incompressible inviscid fluid layers has been studied by various authors, (see, for example [4–6]) the investigation of nonlinear Kelvin–Helmholtz instability of such system is rather scant; however some recent and representative results of related works are contained in [7–11].

The purpose of this paper is to investigate the Kelvin–Helmholtz nonlinear instability of the interface between the vapour and liquid phases of a fluid when there is a mass and heat transfer across the interface.

The multiple scales method [12] which is applied in the recent work of Lee [13, 14] is used to obtain a first-order nonlinear differential equation, and in the present work, we have shown in graphical forms the regions of stability for various values of the flow speed. It is found that the instability is much reduced in the nonlinear theory and the regions of stability exist in the form of a band.

The basic equations with the accompanying boundary conditions are given in section 2. The first-order theory and the linear dispersion relation are obtained in section 3. In section 4 we have derived second-order solutions. In section 5 a first-order nonlinear differential equation is obtained, and in section 6 some numerical examples are presented in graphical form.

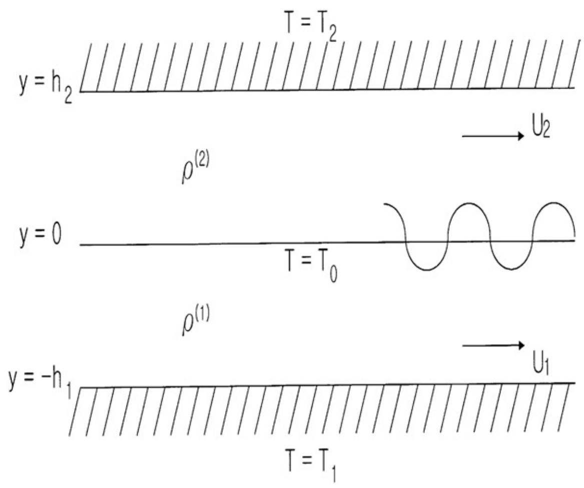

Figure 1. The equilibrium configuration of the fluid system.

2. Formulation of the problem and basic equations

Consider two inviscid, incompressible fluid layers as shown in figure 1. The interface, after a disturbance, is given by the equation

$$F(x, y, t) = y - \eta(x, t) = 0, \tag{2.1}$$

where η is the perturbation from the equilibrium interface $y = 0$, and for which the outward normal vector is written as

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left\{1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right\}^{-1/2} \left(-\frac{\partial \eta}{\partial x}\,\mathbf{e}_x + \mathbf{e}_y\right). \tag{2.2}$$

With the introduction of the function $F(x, y, t) = 0$, equations can be written in more concise forms. The fluid phase 1, of density $\rho^{(1)}$, occupies the region $-h_1 < y < \eta$, and, the fluid phase 2, of density $\rho^{(2)}$, occupies the region $\eta < y < h_2$. The lower and upper fluids are streaming along the x-axis with uniform velocities U_1 and U_2, respectively. We assume that the bounding surfaces $y = -h_1$ and $y = h_2$ are taken as rigid. The temperature at $y = -h_1$, $y = 0$ and $y = h_2$ are taken as T_1, T_0 and T_2 respectively. We assume that the fluid flow is irrotational in the region so that velocity potentials are $\phi^{(1)}$ and $\phi^{(2)}$ for fluid phases 1 and 2. In each fluid phase

$$\nabla^2 \phi^{(j)} = 0, \qquad (j = 1, 2). \tag{2.3}$$

The velocity potentials satisfy the boundary conditions

$$\frac{\partial \phi^{(1)}}{\partial y} = 0 \qquad \text{on} \quad y = -h_1, \tag{2.4}$$

$$\frac{\partial \phi^{(2)}}{\partial y} = 0 \qquad \text{on} \quad y = h_2. \tag{2.5}$$

The interfacial conditions that express the conservation of mass across the interface are given by [6]

$$\left[\!\left[\rho \left(\frac{\partial F}{\partial t} + \nabla \phi \cdot \nabla F\right)\right]\!\right] = 0, \qquad \text{or} \qquad \left[\!\left[\rho \left(\frac{\partial \phi}{\partial y} - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x}\right)\right]\!\right] = 0 \quad \text{at} \quad y = \eta, \tag{2.6}$$

where $[\![h]\!]$ represents the difference in a quantity as we cross the interface, i.e., $[\![h]\!] = h^{(2)} - h^{(1)}$, where superscripts refer to upper and lower fluids, respectively.

The interfacial condition for energy is

$$L\rho^{(1)}\left(\frac{\partial F}{\partial t} + \nabla\phi^{(1)}\cdot\nabla F\right) = S(\eta) \quad \text{at} \quad y = \eta, \tag{2.7}$$

where L is the latent heat released when the fluid is transformed from phase 1 to phase 2. The expression $S(\eta)$ on the right-hand side of (2.7) essentially represents the net heat flux from the interface, while the left-hand side of (2.7) represents the latent heat released during the phase transformation, when such a phase transformation takes place.

In the equilibrium state, the heat fluxes in the positive y-direction in the fluid phases 1 and 2 are $K_1(T_1 - T_0)/h_1$ and $K_2(T_0 - T_2)/h_2$, where K_1 and K_2 are the heat conductivities of the two fluids. As in Hsieh (1978), we denote

$$S(y) = \frac{K_2(T_0 - T_2)}{h_2 - y} - \frac{K_1(T_1 - T_0)}{h_1 + y}. \tag{2.8}$$

When the interface is perturbed to become $y = \eta$, the function S in (2.7) is given by (2.8) and we expand it about $y = 0$ by Maclaurin's expansion, such as

$$S(\eta) = S(0) + \eta S'(0) + \tfrac{1}{2}\eta^2 S''(0) + \tfrac{1}{6}\eta^3 S'''(0) + \cdots, \tag{2.9}$$

and since $S(0) = 0$, from (2.8) we obtain

$$\frac{K_2(T_0 - T_2)}{h_2} = \frac{K_1(T_1 - T_0)}{h_1} = G(\text{say}). \tag{2.10}$$

Thus in the equilibrium state the heat fluxes are equal across the interface in the two fluids.

From (2.1), and (2.7)–(2.10), we have

$$\rho^{(1)}\left(\frac{\partial\phi^{(1)}}{\partial y} - \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial x}\frac{\partial\phi^{(1)}}{\partial x}\right) = \alpha(\eta + \alpha_2\eta^2 + \alpha_3\eta^3), \tag{2.11}$$

where

$$\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right), \qquad \alpha_2 = -\left(\frac{1}{h_1} - \frac{1}{h_2}\right), \qquad \alpha_3 = \frac{h_1^3 + h_2^3}{(h_1 + h_2)h_1^2 h_2^2}$$

By considering the mass transfer across the interface, the conservation of momentum balance is

$$\rho^{(1)}(\nabla\phi^{(1)}\cdot\nabla F)\left(\frac{\partial F}{\partial t} + \nabla\phi^{(1)}\cdot\nabla F\right) = \rho^{(2)}(\nabla\phi^{(2)}\cdot\nabla F)\left(\frac{\partial F}{\partial t} + \nabla\phi^{(2)}\cdot\nabla F\right) + (p_2 - p_1 + \sigma\nabla\cdot\mathbf{n})|\nabla F|^2 \quad \text{at} \quad y = \eta, \tag{2.12}$$

where p is the pressure and σ is the surface tension coefficient, respectively.

We use Bernoulli's equation to eliminate the pressure in the above condition (2.12) which can be rewritten as

$$\left[\!\!\left[\rho\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + g\eta - \left\{1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right\}^{-1}\left(\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y}\right) \times\left(\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y}\right)\right\}\right]\!\!\right] = -\sigma\frac{\partial^2\eta}{\partial x^2}\left\{1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right\}^{-3/2}, \tag{2.13}$$

where g is the gravity.

To investigate the nonlinear effects on the stability of the system, we employ the method of multiple scales. Introducing ϵ as a small parameter, we assume the following expansion of the variables:

$$\eta = \sum_{n=1}^{3} \epsilon^n \eta_n(x, t_0, t_1, t_2) + O(\epsilon^4), \tag{2.14}$$

$$\phi^{(j)} = \sum_{n=0}^{3} \epsilon^n \phi_n^{(j)}(x, y, t_0, t_1, t_2) + O(\epsilon^4), \qquad (j = 1, 2) \tag{2.15}$$

where $t_n = \epsilon^n t (n = 0, 1, 2)$. The quantities appearing in the field equations (2.3) and the boundary conditions (2.6), (2.11) and (2.13) can now be expressed in the Maclaurin series expansion around $y = 0$. Then, we use (2.14) and (2.15) and equate the coefficients of equal power series in ϵ to obtain the linear and the successive nonlinear partial differential equations of various orders (see appendix B).

3. Linear theory

We take

$$\phi_0^{(j)} = U_j x, \qquad (j = 1, 2).$$

The linear wave solutions of (2.3) subject to boundary conditions yield

$$\eta_1 = A(t_1, t_2)\,\mathrm{e}^{\mathrm{i}\theta} + \bar{A}(t_1, t_2)\,\mathrm{e}^{-\mathrm{i}\theta}, \tag{3.1}$$

$$\phi_1^{(1)} = \frac{1}{k}\left(\frac{\alpha}{\rho^{(1)}} - \mathrm{i}\omega + \mathrm{i}kU_1\right) A(t_1, t_2) \frac{\cosh k(y + h_1)}{\sinh kh_1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{3.2}$$

$$\phi_1^{(2)} = -\frac{1}{k}\left(\frac{\alpha}{\rho^{(2)}} - \mathrm{i}\omega + \mathrm{i}kU_2\right) A(t_1, t_2) \frac{\cosh k(y - h_2)}{\sinh kh_2}\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}, \tag{3.3}$$

where $\theta = kx - \omega t_0$.

Remark. In (3.1)–(3.3), and in the following, the complex conjugate of $A\,\mathrm{e}^{\mathrm{i}\theta}$ is denoted by $\bar{A}\,\mathrm{e}^{-\mathrm{i}\theta}$ but not by $\bar{A}\,\mathrm{e}^{-\mathrm{i}\bar{\theta}}$. So the complex conjugate notation here is not rigorous.

Substituting (3.1)–(3.3) into (2.13), we obtain the following dispersion relation:

$$D(\omega, k) = a_0\omega^2 + (a_1 + \mathrm{i}b_1)\omega + a_2 + \mathrm{i}b_2 = 0, \tag{3.4}$$

where

$$\begin{aligned}
a_0 &= \rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2, \\
a_1 &= -2k\{\rho^{(2)} U_2 \coth kh_2 + \rho^{(1)} U_1 \coth kh_1\}, \\
b_1 &= \alpha\{\coth kh_1 + \coth kh_2\}, \\
a_2 &= k^2\left\{\rho^{(1)} U_1^2 \coth kh_1 + \rho^{(2)} U_2^2 \coth kh_2\right\} + (\rho^{(2)} - \rho^{(1)})gk - \sigma k^3, \\
b_2 &= -\alpha k\{U_1 \coth kh_1 + U_2 \coth kh_2\}.
\end{aligned}$$

(i) When $\alpha = 0$, (3.4) reduces to

$$a_0\omega^2 + a_1\omega + a_2 = 0. \tag{3.5}$$

Therefore the system is stable if

$$a_1^2 - 4a_0a_2 > 0, \tag{3.6}$$

or

$$\sigma k^2 - (\rho_2 - \rho_1)g - k\frac{\rho^{(1)}\rho^{(2)}\coth kh_1 \coth kh_2 (U_2 - U_1)^2}{\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2} > 0. \tag{3.7}$$

It is clear from the above inequality that the streaming has a destabilizing effect on the stability of an interface. (ii) when $\alpha \neq 0$, we find that necessary and sufficient stability conditions for (3.4) are

$$b_1 > 0, \tag{3.8}$$

and

$$a_0 b_2^2 - a_1 b_1 b_2 + a_2 b_1^2 < 0, \tag{3.9}$$

since a_0 is always positive.

We note that condition (3.8) is trivially satisfied since α is always positive. Putting the values of a_0, a_1, a_2, b_1 and b_2 from(3.4) into (3.9), it can be shown that the condition for the stability of the system is

$$\sigma k^2 - (\rho_2 - \rho_1)g - k\frac{\rho^{(1)}\rho^{(2)}\coth kh_1 \coth kh_2 (U_2 - U_1)^2}{\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2} \times \left[1 + \frac{\coth kh_1 \coth kh_2 (\rho^{(1)} - \rho^{(2)})^2}{(\coth kh_1 + \coth kh_2)^2 \rho^{(1)}\rho^{(2)}}\right] > 0. \tag{3.10}$$

The stability condition (3.10) differs from (3.7) by the additional last term

$$\frac{\coth kh_1 \coth kh_2 (\rho^{(1)} - \rho^{(2)})^2}{\rho^{(1)}\rho^{(2)}(\coth kh_1 + \coth kh_2)^2}.$$

Thus condition (3.10) is valid for an infinitesimal α and when $\alpha = 0$ the last term is absent. These results seem to be new.

4. Second-order solutions

With the use of the first-order solutions , we obtain the equations for the second-order problem

$$\nabla_0^2 \phi_2^{(i)} = 0, \qquad (i = 1, 2) \tag{4.1}$$

and the boundary conditions at $y = 0$.

$$\rho^{(1)}\left\{\frac{\partial \phi_2^{(1)}}{\partial y} - \frac{\partial \eta_2}{\partial t_0} - \frac{\partial \eta_2}{\partial x}U_1\right\} - \alpha\eta_2 = \left[-\rho^{(1)}\left\{\frac{\alpha}{\rho^{(1)}} - \mathrm{i}\omega + \mathrm{i}U_1 k\right\} 2k\coth kh_1 + \alpha\alpha_2\right] \times A^2\,\mathrm{e}^{2\mathrm{i}\theta} - \rho^{(1)}\frac{\partial A}{\partial t_1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} - 2\alpha\alpha_2|A|^2, \tag{4.2}$$

$$\rho^{(2)}\left\{\frac{\partial \phi_2^{(2)}}{\partial y} - \frac{\partial \eta_2}{\partial t_0} - \frac{\partial \eta_2}{\partial x}U_2\right\} - \alpha\eta_2 = \left[\rho^{(2)}\left\{\frac{\alpha}{\rho^{(2)}} - \mathrm{i}\omega + \mathrm{i}U_2 k\right\} 2k\coth kh_2 + \alpha\alpha_2\right] \times A^2\,\mathrm{e}^{2\mathrm{i}\theta} - \rho^{(2)}\frac{\partial A}{\partial t_1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} - 2\alpha\alpha_2|A|^2, \tag{4.3}$$

$$\rho^{(2)}\frac{\partial \phi_2^{(2)}}{\partial y} - \rho^{(1)}\frac{\partial \phi_2^{(1)}}{\partial y} - \{\rho^{(2)} - \rho^{(1)}\}\frac{\partial \eta_2}{\partial t_0} - \{\rho^{(2)}U_2 - \rho^{(1)}U_1\}\frac{\partial \eta_2}{\partial x}$$
$$= 2k\left\{\rho^{(2)}\left(\frac{\alpha}{\rho^{(2)}} - \mathrm{i}\omega + \mathrm{i}kU_2\right)\coth kh_2 + \rho^{(1)}\left(\frac{\alpha}{\rho^{(1)}} - \mathrm{i}\omega + \mathrm{i}kU_1\right)\coth kh_1\right\} A^2\,\mathrm{e}^{2\mathrm{i}\theta} + \{\rho^{(2)} - \rho^{(1)}\}\frac{\partial A}{\partial t_1}\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} \tag{4.4}$$

$$\rho^{(2)}\left\{\frac{\partial\phi_2^{(2)}}{\partial t_0}+U_2\frac{\partial\phi_2^{(2)}}{\partial x}\right\}-\rho^{(1)}\left\{\frac{\partial\phi_2^{(1)}}{\partial t_0}+U_1\frac{\partial\phi_2^{(1)}}{\partial x}\right\}+\sigma\frac{\partial^2\eta_2}{\partial x^2}+g\eta_2(\rho^{(2)}-\rho^{(1)})$$

$$=\left[\!\!\left[\rho\{2\omega(\omega-kU)-\frac{1}{2}(\omega-kU)^2(1+\coth^2 kh)\}+\frac{\alpha^2}{2\rho}(1+\coth^2 kh)\right.\right.$$

$$\left.\left.-\mathrm{i}\alpha(\omega-kU)(1+\coth^2 kh)-k\rho U\left\{\mathrm{i}\frac{3\alpha}{\rho}+2(\omega-kU)\right\}\right]\!\!\right]A^2\,\mathrm{e}^{2\mathrm{i}\theta}$$

$$+\frac{1}{k}\left\{\rho^{(1)}\left(\frac{\alpha}{\rho^{(1)}}-\mathrm{i}\omega+\mathrm{i}kU_1\right)\coth kh_1\right.$$

$$\left.+\rho^{(2)}\left(\frac{\alpha}{\rho^{(2)}}-\mathrm{i}\omega+\mathrm{i}kU_2\right)\coth kh_2\right\}\frac{\partial A}{\partial t_1}\,\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}$$

$$+\left[\!\!\left[\rho\left\{\frac{\alpha^2}{\rho^2}+(\omega-kU)^2\right\}(1-\coth^2 kh)\right]\!\!\right]|A|^2. \tag{4.5}$$

The non-secularity condition for the existence of the uniformly valid solution is

$$\frac{\partial A}{\partial t_1}=0. \tag{4.6}$$

Equations (4.1) to (4.5) furnish the second-order solutions

$$\eta_2=-2\alpha_2|A|^2+A_2A^2\,\mathrm{e}^{2\mathrm{i}\theta}+\bar{A}_2\bar{A}^2\,\mathrm{e}^{-2\mathrm{i}\theta}, \tag{4.7}$$

$$\phi_2^{(1)}=B_2^{(1)}A^2\,\mathrm{e}^{2\mathrm{i}\theta}\frac{\cosh 2k(y+h_1)}{\sinh 2kh_1}+\text{c.c.}+b^{(1)}(t_0,t_1,t_2), \tag{4.8}$$

$$\phi_2^{(2)}=-B_2^{(2)}A^2\mathrm{e}^{2\mathrm{i}\theta}\frac{\cosh 2k(y-h_2)}{\sinh 2kh_2}+\text{c.c.}+b^{(2)}(t_0,t_1,t_2). \tag{4.9}$$

Symbols in the above equations are found in appendix A.

5. Third-order solutions

We examine now the third order problem

$$\nabla_0^2\phi_3^{(i)}=0,\qquad (i=1,2). \tag{5.1}$$

On substituting the values of η_1, $\phi_1^{(i)}$ from (3.1)–(3.3) and η_2, $\phi_2^{(i)}$ from (4.7)–(4.9) into (B.8), we obtain

$$\phi_3^{(1)}=\left\{C_3^{(1)}A^2\bar{A}+\frac{1}{k}\frac{\partial A}{\partial t_2}\right\}\frac{\cosh k(y+h_1)}{\sinh kh_1}\,\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}, \tag{5.2}$$

$$\phi_3^{(2)}=\left\{C_3^{(2)}A^2\bar{A}-\frac{1}{k}\frac{\partial A}{\partial t_2}\right\}\frac{\cosh k(y-h_2)}{\sinh kh_2}\,\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.} \tag{5.3}$$

Symbols in the above equations are found in appendix A.

We substitute the first- and second-order solutions into the third-order equation. In order to avoid non-uniformity of the expansion, we again impose the condition that secular terms vanish. Then from (B.9), we find

$$\frac{\mathrm{i}}{k}\frac{\partial D}{\partial\omega}\frac{\partial A}{\partial t_2}+qA^2\bar{A}=0, \tag{5.4}$$

where q is defined in appendix A.

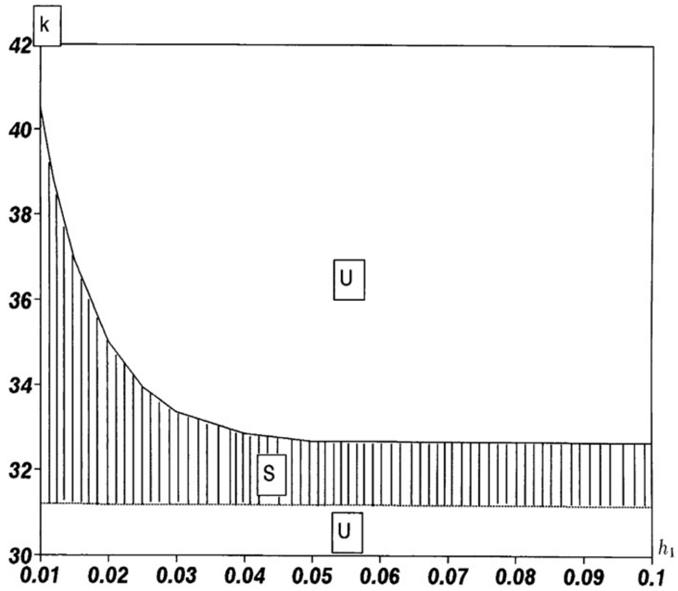

Figure 2. Stability diagram for $U_2 = 0.5$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

We rewrite (5.4) as

$$\frac{\partial A}{\partial t_2} + QA^2\bar{A} = 0, \tag{5.5}$$

which can be easily integrated as

$$|A|^2 = \frac{1}{|A_0|^{-2} + 2(\mathrm{Re}\, Q)t_2}, \tag{5.6}$$

where A_0 is the initial amplitude and Re Q means the real part of Q. From (5.6), the stability condition is

$$\mathrm{Re}\, Q > 0. \tag{5.7}$$

6. Numerical results

The stability of the system depends on condition (5.7). We show the stable and unstable regions in figures 2–9. In these figures, the dotted lines (lower lines) represent the linear curve which divides the plane into a stable region (above the curve) and an unstable region (below the curve). The upper curve (solid lines) is a nonlinear boundary above which the flow is unstable and below which the flow is stable. The region above the solid line (Labelled by U) was originally a stabilized region by the linear theory but is now a destabilized region by the nonlinear theory since in this region the amplitude will blow up as time passes. The nonlinearly stable regions occupy far below dotted line (linear curve), but below the linear curve, even though the amplitude is bounded, the exponential function grows to infinity as time passes, so we have to exclude these regions from the stable regions. Therefore, only the overlapping shaded regions S (by vertical lines) are the nonlinearly stabilized regions. Since below the dotted line which is the linear linear curve (lower curve) the region is unstable, it is labelled by U. The unshaded regions (intact regions) are unstable regions and labelled by U.

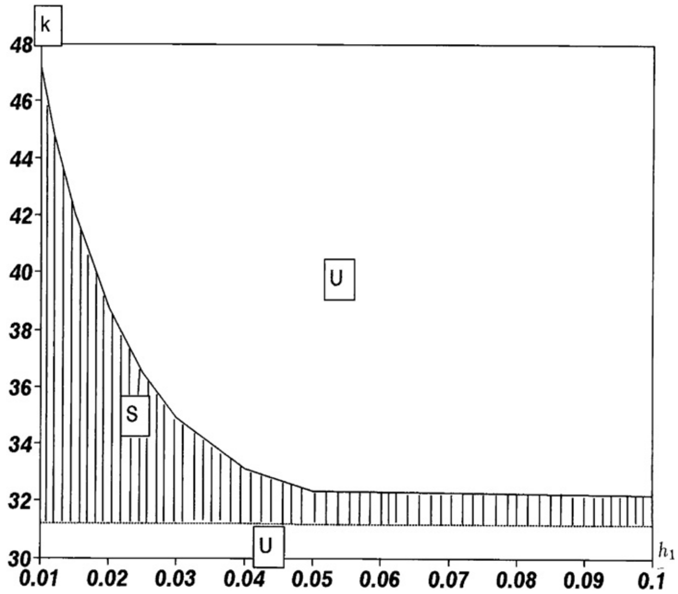

Figure 3. Stability diagram for $U_2 = 0.5$ cm s^{-1} and $\alpha = 1.5$ gm cm^{-3} s^{-1}.

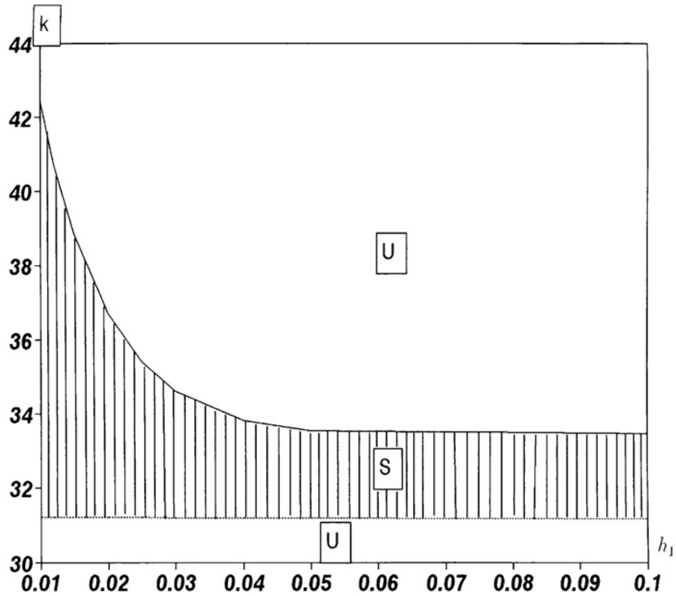

Figure 4. Stability diagram for $U_2 = 1$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

Figures 2 and 4 show the stability diagrams in the h_1–k–plane corresponding to the cases $\alpha = 1$ gm cm^{-3} s^{-1}, $\sigma = 0.06$ dyn cm^{-1}, $\rho_1 = 3.652 \times 10^{-4}$ gm cm^{-3}, $\rho_2 = 5.97 \times 10^{-2}$ gm cm^{-3}, $g = 980$ cm s^{-2}.

In figures 3 and 5, α is chosen to be 1.5 gm cm^{-3} s $^{-1}$; other values remain the same.

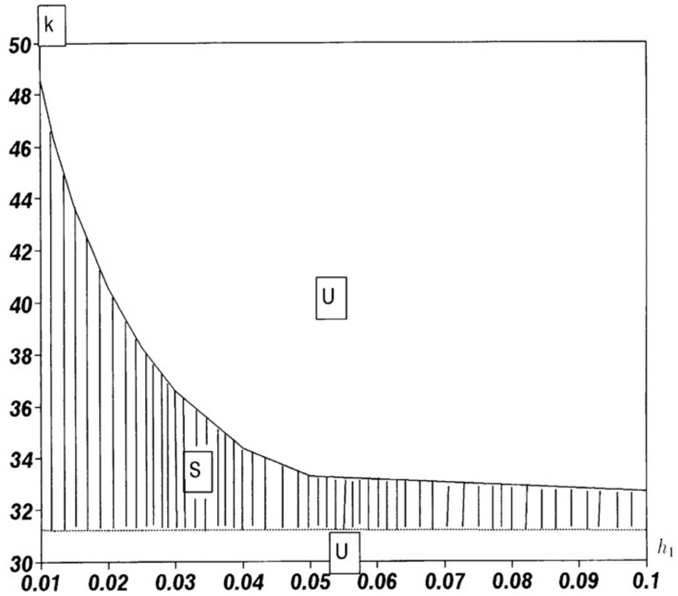

Figure 5. Stability diagram for $U_2 = 1$ cm s^{-1} and $\alpha = 1.5$ gm cm^{-3} s^{-1}.

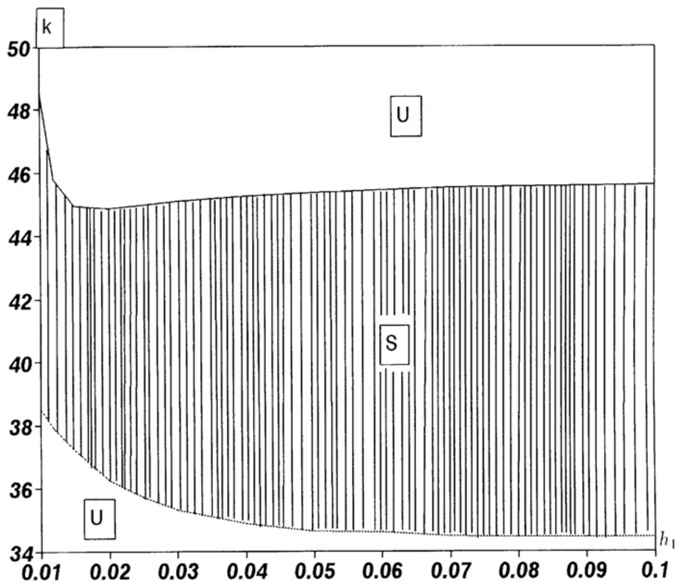

Figure 6. Stability diagram for $U_2 = 5$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

Unlike the linear theory, stability regions form narrow bands, thus in the nonlinear theory the region of stability is much reduced. In these examples, we have taken $U_1 = 0$. In figures 2 and 3, U_2 is equal to 0.5 cm s^{-1}. In figures 4 and 5 $U_2 = 1$ cm s^{-1}. From these figures, we see that in the nonlinear theory, the regions for which the system is stable are larger for the stronger heat flux.

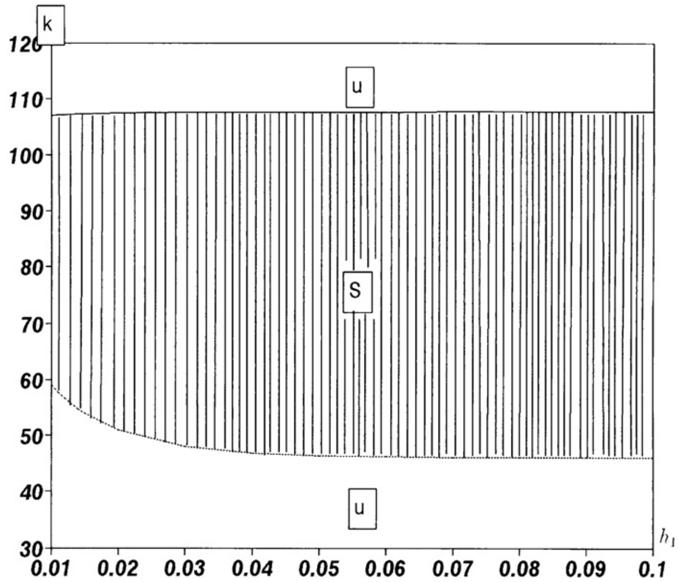

Figure 7. Stability diagram for $U_2 = 10$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

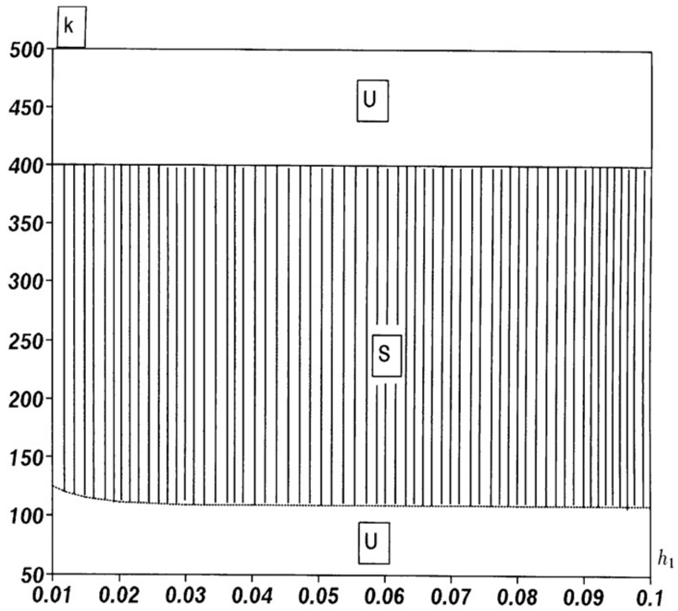

Figure 8. Stability diagram for $U_2 = 20$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

For larger values of velocities, stability regions exist only for very large values of k and are displayed in figures 6–9 for $U_2 = 5$ cm s^{-1}–500 cm s^{-1}.

In figures 2–9, the thickness of vapour varies from 0.01 cm to 0.1 cm, and $h_2 = (2 - h_1)$ cm. From these figures we can see that when the fluid flow is quasi-static, the

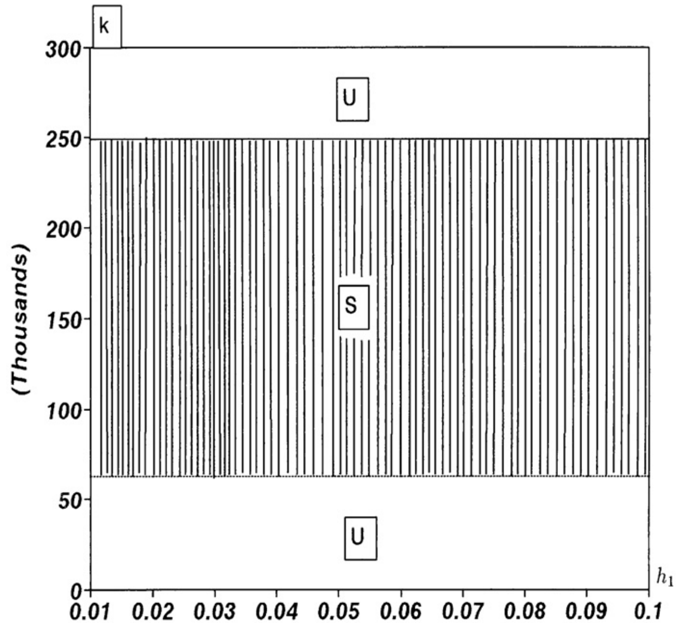

Figure 9. Stability diagram for $U_2 = 500$ cm s^{-1} and $\alpha = 1$ gm cm^{-3} s^{-1}.

region of stability reduces as the thickness of the vapour increases. So, with the same heat flux, the thinner the vapour layer, the easier the system can be stabilized.

However, figures 8 and 9 show that when the streaming velocity is very large, the film thickness has little influence on the stability of the system due to the fact that the equation is now dominated by the effect of velocity. To sum up, we have presented an analysis of nonlinear Kelvin–Helmholtz instability. It is found that unlike in linear theory, the stability region is existent in the form of narrow bands, the width of which, when the flow is quasi-static, reduces with the increment of the thickness of the vapour layer. The region of stability is enlarged with the stronger heat flux.

7. Conclusions

The stability of the Kelvin–Helmholtz flow when there is mass and heat transfer across the interface is studied. Using the method of multiple time scales, a first-order nonlinear differential equation describing the evolution of nonlinear waves is obtained. Unlike linear theory, with nonlinear theory it is evident that the mass and heat transfer plays an important role in the stability of fluid, in a situation like film boiling and the results can be summarized as follows.

1. Stronger heat flux enlarges the region of stability.
2. When the streaming velocity is very small, the thinner the vapour layer, the more easily the system can be stabilized whereas when the velocity is very large, the stability is little influenced by the thickness of vapour layer.
3. With higher velocities the stability takes place at larger wave numbers.

Appendix A

Symbols in (4.7)–(4.9) are

$$A_2 = \frac{2k}{D(2\omega, 2k)} \left\{ \left[\!\!\left[\frac{\rho \mathrm{i}(\omega - kU)}{k} \beta \coth 2kh + \frac{\rho}{2}(\coth^2 kh + 1) \left(\frac{\alpha}{\rho} - \mathrm{i}\omega + \mathrm{i}kU \right)^2 \right.\right.\right.$$
$$\left.\left.\left. + 2\rho(\omega - kU)^2 - \mathrm{i}3\alpha kU \right]\!\!\right] - \mathrm{i}\frac{\alpha\alpha_2}{k}[(\omega - kU_1)\coth 2kh_1 \right.$$
$$\left. + (\omega - kU_2)\coth 2kh_2] \right\}, \tag{A.1}$$

$$B_2^{(1)} = \frac{1}{2k}\left[\beta^{(1)} + \left\{ \frac{\alpha}{\rho^{(1)}} - 2\mathrm{i}(\omega - kU_1) \right\} A_2 + \frac{\alpha\alpha_2}{\rho^{(1)}} \right], \tag{A.2}$$

$$B_2^{(2)} = \frac{1}{2k}\left[-\beta^{(2)} + \left\{ \frac{\alpha}{\rho^{(2)}} - 2\mathrm{i}(\omega - kU_2) \right\} A_2 + \frac{\alpha\alpha_2}{\rho^{(2)}} \right], \tag{A.3}$$

$$\beta^{(j)} = -2k\left\{ \frac{\alpha}{\rho^{(j)}} - i(\omega - kU_j) \right\} \coth kh_j \qquad (j = 1, 2), \tag{A.4}$$

$$\rho^{(2)}\frac{\partial b^{(2)}}{\partial t_0} - \rho^{(1)}\frac{\partial b^{(1)}}{\partial t_0} = \left[\!\!\left[\rho\left\{ \frac{\alpha^2}{\rho^2} + (\omega - kU)^2 \right\}(1 - \coth^2 kh) + 2\rho g\alpha_2 \right]\!\!\right] |A|^2, \tag{A.5}$$

Symbols in (5.2)–(5.3) are

$$C_3^{(1)} = -k\left[2B_2^{(1)}\coth 2kh_1 - 2\coth kh_1\left(\frac{\alpha}{\rho^{(1)}} - \mathrm{i}\omega + \mathrm{i}kU_1 \right)\frac{\alpha_2}{k} \right.$$
$$+ \frac{1}{2}\left\{ \frac{\alpha}{\rho^{(1)}} - 3\mathrm{i}(\omega - kU_1) \right\} + \frac{\alpha}{\rho^{(1)}k^2}\left(4\alpha_2^2 - 3\alpha_3\right)$$
$$\left. - \left\{ \left(\frac{\alpha}{\rho^{(1)}} + \mathrm{i}\omega - \mathrm{i}kU_1 \right)\coth kh + \frac{2\alpha\alpha_2}{\rho^{(1)}k} \right\}\frac{A_2}{k} \right], \tag{A.6}$$

$$C_3^{(2)} = -k\left[2B_2^{(2)}\coth 2kh_2 - 2\coth kh_2\left(\frac{\alpha}{\rho^{(2)}} - \mathrm{i}\omega + \mathrm{i}kU_2 \right)\frac{\alpha_2}{k} \right.$$
$$- \frac{1}{2}\left\{ \frac{\alpha}{\rho^{(2)}} - 3\mathrm{i}(\omega - kU_2) \right\} - \frac{\alpha}{\rho^{(2)}k^2}\left(4\alpha_2^2 - 3\alpha_3\right)$$
$$\left. + \left\{ -\left(\frac{\alpha}{\rho^{(2)}} + \mathrm{i}\omega - \mathrm{i}kU_2 \right)\coth kh + \frac{2\alpha\alpha_2}{\rho^{(2)}k} \right\}\frac{A_2}{k} \right], \tag{A.7}$$

q in (5.4) is

$$q = \left[\!\!\left[\rho\left(-\mathrm{i}(\omega - kU)C_3\coth kh + A_2\left\{ \mathrm{i}\frac{\alpha}{\rho}(-\omega + 3kU) + (\omega - kU)(\omega + kU) \right\} \right.\right.\right.$$
$$+ B_2 2k\left\{ \frac{\alpha}{\rho}(\coth kh\coth 2kh - 1) + \mathrm{i}[-2\omega + kU + (\omega - kU)\coth kh\coth 2kh] \right\}$$
$$\left.\left.\left. + \mathrm{i}\alpha_2 2(\omega - kU)\left\{ \frac{\alpha}{\rho} - \mathrm{i}\omega + \mathrm{i}kU \right\} \right) \right]\!\!\right]$$

$$-k\left(\rho^{(1)}\coth kh_1\left\{\frac{1}{2}(\omega-kU_1)(\omega-5kU_1)+\mathrm{i}\frac{\alpha}{2\rho^{(1)}}(3\omega-7kU_1)\right\}\right.$$
$$\left.+\rho^{(2)}\coth kh_2\left\{\frac{1}{2}(\omega-kU_2)(\omega-5kU_2)+\mathrm{i}\frac{\alpha}{2\rho^{(2)}}(3\omega-7kU_2)\right\}\right)+\sigma\frac{3}{2}k^4. \tag{A.8}$$

Appendix B

The interfacial conditions are given on $r = R$ as

order $O(\epsilon)$

$$\left[\!\!\left[\rho\left(\frac{\partial\phi_1}{\partial y}-\frac{\partial\eta_1}{\partial t_0}-\frac{\partial\eta_1}{\partial x}\frac{\partial\phi_0}{\partial x}\right)\right]\!\!\right]=0, \tag{B.1}$$

$$\rho^{(1)}\left(\frac{\partial\phi_1^{(1)}}{\partial y}-\frac{\partial\eta_1}{\partial t_0}-\frac{\partial\eta_1}{\partial x}\frac{\partial\phi_0}{\partial x}\right)=\alpha\eta_1, \tag{B.2}$$

$$\left[\!\!\left[\rho\left(\frac{\partial\phi_1}{\partial t_0}+g\eta_1+\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_0}{\partial x}\right)\right]\!\!\right]=-\sigma\frac{\partial^2\eta_1}{\partial x^2}, \tag{B.3}$$

order $O(\epsilon^2)$

$$\left[\!\!\left[\rho\left(\frac{\partial\phi_2}{\partial y}+\frac{\partial^2\phi_1}{\partial y^2}\eta_1-\frac{\partial\eta_2}{\partial t_0}-\frac{\partial\eta_1}{\partial t_1}-\frac{\partial\eta_1}{\partial x}\frac{\partial\phi_1}{\partial x}-\frac{\partial\eta_2}{\partial x}\frac{\partial\phi_0}{\partial x}\right)\right]\!\!\right]=0, \tag{B.4}$$

$$\rho^{(1)}\left(\frac{\partial\phi_2^{(1)}}{\partial y}+\frac{\partial^2\phi_1^{(1)}}{\partial y^2}\eta_1-\frac{\partial\eta_2}{\partial t_0}-\frac{\partial\eta_1}{\partial t_1}-\frac{\partial\eta_1}{\partial x}\frac{\partial\phi_1^{(1)}}{\partial x}-\frac{\partial\eta_2}{\partial x}\frac{\partial\phi_0}{\partial x}\right)=\alpha\left(\eta_2+\alpha_2\eta_1^2\right), \tag{B.5}$$

$$\left[\!\!\left[\rho\left\{\frac{\partial\phi_2}{\partial t_0}+\frac{\partial\phi_1}{\partial t_1}+\frac{\partial^2\phi_1}{\partial t_0\partial y}\eta_1+\frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2+\left(\frac{\partial\phi_1}{\partial y}\right)^2\right]+\frac{\partial\phi_2}{\partial x}\frac{\partial\phi_0}{\partial x}+\frac{\partial\phi_1}{\partial y}\left(\frac{\partial\eta_1}{\partial t_0}-\frac{\partial\phi_1}{\partial y}\right)\right.\right.\right.$$
$$\left.\left.\left.+g\eta_2+\frac{\partial\phi_0}{\partial x}\left(-\frac{\partial\eta_1}{\partial x}\frac{\partial\eta_1}{\partial t_0}+2\frac{\partial\eta_1}{\partial x}\frac{\partial\phi_1}{\partial y}-\left(\frac{\partial\eta_1}{\partial x}\right)^2\frac{\partial\phi_0}{\partial x}+\frac{\partial^2\phi_1}{\partial x\partial y}\eta_1\right)\right\}\right]\!\!\right]$$
$$=-\sigma\frac{\partial^2\eta_2}{\partial x^2}, \tag{B.6}$$

order $O(\epsilon^3)$

$$\left[\!\!\left[\rho\left\{\frac{\partial\phi_3}{\partial y}+\frac{\partial^2\phi_2}{\partial y^2}\eta_1+\frac{\partial^2\phi_1}{\partial y^2}\eta_2+\frac{1}{2}\frac{\partial^3\phi_1}{\partial y^3}\eta_1^2-\frac{\partial\eta_3}{\partial t_0}-\frac{\partial\eta_2}{\partial t_1}-\frac{\partial\eta_1}{\partial t_2}\right.\right.\right.$$
$$\left.\left.\left.-\frac{\partial\eta_1}{\partial x}\left(\frac{\partial\phi_2}{\partial x}+\frac{\partial^2\phi_1}{\partial x\partial y}\eta_1\right)-\frac{\partial\eta_2}{\partial x}\frac{\partial\phi_1}{\partial x}-\frac{\partial\eta_3}{\partial x}\frac{\partial\phi_0}{\partial x}\right\}\right]\!\!\right]=0, \tag{B.7}$$

$$\rho^{(1)}\left\{\frac{\partial\phi_3^{(1)}}{\partial y}+\frac{\partial^2\phi_2^{(1)}}{\partial y^2}\eta_1+\frac{\partial^2\phi_1^{(1)}}{\partial y^2}\eta_2+\frac{1}{2}\frac{\partial^3\phi_1^{(1)}}{\partial y^3}\eta_1^2-\frac{\partial\eta_3}{\partial t_0}-\frac{\partial\eta_2}{\partial t_1}-\frac{\partial\eta_1}{\partial t_2}\right.$$
$$\left.-\frac{\partial\eta_1}{\partial x}\left(\frac{\partial\phi_2^{(1)}}{\partial x}+\frac{\partial^2\phi_1^{(1)}}{\partial x\partial y}\eta_1\right)-\frac{\partial\eta_2}{\partial x}\frac{\partial\phi_1^{(1)}}{\partial x}-\frac{\partial\eta_3}{\partial x}\frac{\partial\phi_0}{\partial x}\right\}$$
$$=\alpha\left(\eta_3+2\alpha_2\eta_1\eta_2+\alpha_3\eta_1^3\right), \tag{B.8}$$

$$\begin{aligned}
\bigg[\!\!\bigg[\rho\bigg\{&\frac{\partial\phi_3}{\partial t_0}+g\eta_3+\frac{\partial\phi_2}{\partial t_1}+\frac{\partial\phi_1}{\partial t_2}+\frac{\partial^2\phi_1}{\partial t_0\partial y}\eta_2+\left(\frac{\partial^2\phi_1}{\partial t_1\partial y}+\frac{\partial^2\phi_2}{\partial t_0\partial y}\right)\eta_1\\
&+\frac{1}{2}\frac{\partial^3\phi_1}{\partial t_0\partial y^2}\eta_1^2+\frac{\partial\phi_1}{\partial y}\left(\frac{\partial\phi_2}{\partial y}+\frac{\partial^2\phi_1}{\partial y^2}\eta_1\right)+\frac{\partial\phi_1}{\partial x}\left(\frac{\partial\phi_2}{\partial x}+\frac{\partial^2\phi_1}{\partial y\partial x}\eta_1\right)\\
&-\left(\frac{\partial\phi_1}{\partial y}-\frac{\partial\eta_1}{\partial t_0}\right)\left(\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}\right)+\frac{\partial\phi_1}{\partial y}\left(\frac{\partial\eta_2}{\partial t_0}+\frac{\partial\eta_1}{\partial t_1}-\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}\right)\\
&+\eta_1\frac{\partial^2\phi_1}{\partial y^2}\left(\frac{\partial\eta_1}{\partial t_0}-2\frac{\partial\phi_1}{\partial y}\right)+\frac{\partial\phi_0}{\partial x}\left(\frac{\partial\phi_3}{\partial x}+\frac{\partial^2\phi_1}{\partial y\partial x}\eta_2+\frac{\partial^2\phi_2}{\partial y\partial x}\eta_1+\frac{\partial^3\phi_1}{\partial y^2\partial x}\frac{\eta_1^2}{2}\right.\\
&+2\frac{\partial\phi_2}{\partial y}\frac{\partial\eta_1}{\partial x}+2\frac{\partial\phi_1}{\partial y}\frac{\partial\eta_2}{\partial x}+2\frac{\partial^2\phi_1}{\partial y^2}\frac{\partial\eta_1}{\partial x}\eta_1-\frac{\partial\eta_2}{\partial t_0}\frac{\partial\eta_1}{\partial x}-\frac{\partial\eta_1}{\partial t_0}\frac{\partial\eta_2}{\partial x}-\frac{\partial\eta_1}{\partial t_1}\frac{\partial\eta_1}{\partial x}\\
&\left.-2\frac{\partial\eta_1}{\partial x}\frac{\partial\eta_2}{\partial x}\frac{\partial\phi_0}{\partial x}\right)-2\left(\frac{\partial\eta_1}{\partial x}\right)^2\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_0}{\partial x}\bigg\}\bigg]\!\!\bigg]=-\sigma\left(\frac{\partial^2\eta_3}{\partial x^2}-\frac{3}{2}\frac{\partial^2\eta_1}{\partial x^2}\right),
\end{aligned}\tag{B.9}$$

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